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About boundedness of stable sequences

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Abstract

A sufficient condition under which a symmetric α-stable process $\{X(n), n \in \mathbf{N}\}$ is a.s. bounded is given. We also show that in some sense this condition is optimal.

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1. Introduction

Let $X(n)$, $n = 1, 2, \ldots$ be symmetric α-stable ($S\alpha S$) process, $0 < \alpha < 2$, given in the form

$$X(n) = \int_E f_n(v) M(\mathrm{d}v), \quad n = 1, 2, \ldots, \tag{1}$$

where M is an $S\alpha S$ random measure with control measure μ and $f_n \in L^\alpha(E, \mathscr{E}, \mu)$. It is known that every $S\alpha S$ process with parameter space $\mathbf{N}$ can be so represented, see Samorodnitsky and Taqqu (1994, p. 565). Many properties of the process are described in terms of the functions f_n. In particular, the necessary condition for a.s. boundedness is

$$\sup_n |f_n(\cdot)| \in L^\alpha(E, \mathscr{E}, \mu). \tag{2}$$

If $0 < \alpha < 1$, then (2) is also sufficient for a.s. boundedness, but in the case $1 \leqslant \alpha < 2$ it is not true. For the latter range of α sufficient conditions are given in terms of metric

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entropy. The reader can find these facts with proofs and discussions in Chapters 10 and 12 of the book by Samorodnitsky and Taqqu (1994).

In this paper we give, in terms of f_n, a simple sufficient condition for sample boundedness of the process (1), where $1 \leqslant \alpha < 2$, and show that in some sense, this condition is the best possible. In what follows C and c (with or without index) are generic positive constants whose values will be allowed to change from line to line.

2. Result

Theorem 1. (i) *Suppose $1 \leqslant \alpha < 2$, the process $X(n)$ is given in form (1) and*

$$\sup_n (\log n)^{1-1/\alpha} |f_n(\cdot)| := q(\cdot) \in L^\alpha(E, \mathscr{E}, \mu) \quad \textit{if } 1 < \alpha < 2,$$

$$\sup_n (\log\log n) |f_n(\cdot)| := q(\cdot) \in L^1(E, \mathscr{E}, \mu) \quad \textit{if } \alpha = 1. \tag{3}$$

Then this process is a.s. bounded.

(ii) *For every $1 \leqslant \alpha < 2$ there exists an a.s. unbounded process $X(n)$ given in form (1) such that*

$$\sup_n (\log n)^{\tau(1-1/\alpha)} |f_n(\cdot)| \in L^\alpha(E, \mathscr{E}, \mu) \quad \textit{if } 1 < \alpha < 2,$$

$$\sup_n (\log\log n)^{\tau} |f_n(\cdot)| \in L^1(E, \mathscr{E}, \mu) \quad \textit{if } \alpha = 1 \tag{4}$$

for each $\tau \in (0, 1)$.

3. Proof

(i) The proof is based on the series representation of stable processes. For $1 \leqslant \alpha < 2$ and the function $q(\cdot)$ defined by (3) put

$$g(v) = \begin{cases} Cq(v)^\alpha & \text{if } q(v) \neq 0, \\ p(v) & \text{if } q(v) = 0, \end{cases} \tag{5}$$

where $p \in L^1(E, \mathscr{E}, \mu)$ is a positive function. One can choose p and C such that

$$\|g\|_{L^1(E,\mathscr{E},\mu)} = 1.$$

Putting

$$\mu_0(A) = \int_E g(v)\mu(\mathrm{d}v), \quad A \in \mathscr{E},$$

we obtain a probabilistic measure and a representation

$$\{X(n), n \in \mathbf{N}\} \overset{\mathrm{d}}{=} \left\{ C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} f_n^*(V_j), n \in \mathbf{N} \right\}, \tag{6}$$

where

$$f_n^*(v) = g(v)^{-1/\alpha} f_n(v), \tag{7}$$

the sequences $\{\varepsilon_j\}$, $\{\Gamma_j\}$ and $\{V_j\}$ are independent, the first of them is a sequence of Rademacher random variables, the second is a sequence of arrival times of Poisson process with unit rate, $\{V_j\}$ are i.i.d. E-valued random elements with joint distribution μ_0, and

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x \, \mathrm{d}x\right)^{-1}.$$

See Samorodnitsky and Taqqu (1994, Section 3.10). It follows from (3) and (5) that

$$|f_n^*(v)| \leqslant \begin{cases} C_1(\log n)^{(1/\alpha)-1} & \text{if } 1 < \alpha < 2, \\ C_1(\log\log n)^{-1} & \text{if } \alpha = 1. \end{cases} \tag{8}$$

To prove a.s. boundedness of the sequence $X(n)$ it is enough to show that

$$\psi(u) := P\left(\sup_n |X(n)| > u\right) \to 0 \quad \text{as } u \to \infty. \tag{9}$$

For a fixed $\delta > 0$ put

$$A_\delta = \bigcup_{j=1}^\infty (\Gamma_j < \delta j). \tag{10}$$

Then

$$\psi(u) \leqslant P(A_\delta) + P\left(\sup_n |X(n)| \mathbf{1}_{A_\delta^c} > u\right). \tag{11}$$

Since

$$P(\Gamma_j \leqslant t) \leqslant \frac{t^j}{j!}$$

we have using Stirling's formula

$$P(A_\delta) \leqslant \sum_{j=1}^\infty P(\Gamma_j \leqslant \delta j) \leqslant \sum_{j=1}^\infty \frac{(\delta j)^j}{j!} \leqslant B \sum_{j=1}^\infty (\delta e)^j = \frac{B\,\delta e}{1-\delta e} \tag{12}$$

for $\delta < \mathrm{e}^{-1}$, where B is an absolute constant.

Further, (6) yields

$$P\left(\sup_n |X(n)| \mathbf{1}_{A_\delta^c} > u\right) \leqslant \sum_{n=1}^\infty P\left(\left|\sum_{j=1}^\infty \varepsilon_j \Gamma_j^{-1/\alpha} f_n^*(V_j)\right| \mathbf{1}_{A_\delta^c} > u\right) := \sum_{n=1}^\infty p(n). \tag{13}$$

Because $\Gamma_j \geqslant \delta j$ for all j on A_δ^c, the contraction principle and (8) give us

$$p(n) \leqslant \begin{cases} 4P\left(\left|\sum_{j=1}^{\infty} \varepsilon_j j^{-1/\alpha}\right| > c\,\delta^{1/\alpha} u(\log n)^{1-1/\alpha}\right) & \text{if } 1 < \alpha < 2, \\ 4P\left(\left|\sum_{j=1}^{\infty} \varepsilon_j j^{-1}\right| > c\,\delta u \log\log n\right) & \text{if } \alpha = 1. \end{cases} \tag{14}$$

It is well known (see Montgomery-Smith, 1990, p. 518) that for all $t > 0$

$$P\left(\left|\sum_{j=1}^{\infty} \varepsilon_j j^{-1/\alpha}\right| > t\right) \leqslant b^{-1}\exp(-bt^{\alpha/(\alpha-1)}) \quad \text{if } 1 < \alpha < 2,$$

$$P\left(\left|\sum_{j=1}^{\infty} \varepsilon_j j^{-1}\right| > t\right) \leqslant b^{-1}\exp(-\exp(bt)) \qquad \text{if } \alpha = 1,$$

where $b = b(\alpha)$ is an absolute constant. From here and (14)

$$p_n(u) \leqslant b^{-1}\exp(-bc^{\alpha/(\alpha-1)}\delta^{1/(\alpha-1)}u^{\alpha/(\alpha-1)}(\log n)^{\alpha/(\alpha-1)(1-1/\alpha)})$$

$$= b^{-1} n^{-bc^{\alpha/(\alpha-1)}\delta^{1/(\alpha-1)}u^{\alpha/(\alpha-1)}}$$

for $1 < \alpha < 2$, and

$$p_n(u) \leqslant b^{-1}\exp(-\exp(bc\,\delta u \log\log n)) = b^{-1}\exp(-[\log n]^{bc\,\delta u})$$

for $\alpha = 1$. These estimates imply that $\sum_{n=1}^{\infty} p_n(u) < \infty$ for u large enough, and that

$$\sum_{n=1}^{\infty} p_n(u) \to 0 \quad \text{as } u \to \infty,$$

which together with (13), (11) and (12) yields

$$\limsup_{u\to\infty} \psi(u) \leqslant \frac{B\,\delta e}{1 - \delta e}.$$

Letting $\delta \to 0$ we get (9) and complete the proof of the first part of the theorem.

(ii) *The case* $1 < \alpha < 2$. We now use some ideas from Samorodnitsky and Taqqu (1994) (see Example 10.4.1, p. 461). Fix $\gamma > 1$ and denote

$$C(\gamma) = \sum_{j=1}^{\infty} \frac{1}{j[\log(j+1)]^{\gamma}}, \qquad a_0 = 0,$$

$$a_k = \frac{1}{C(\gamma)} \sum_{j=1}^{k} \frac{1}{j[\log(j+1)]^{\gamma}}, \qquad k = 1, 2, \ldots\ . \tag{15}$$

For $m = 1, 2, \ldots$ let U_m be the set of all m-dimensional vectors of the type $(\pm 1, \ldots, \pm 1)$, $V_m = \{n \in \mathbf{N} : 2^m \leqslant n < 2^{m+1}\}$ and $\phi_m : V_m \to U_m$ be a one-to-one map. Now we define functions $f_n(v), 0 \leqslant v \leqslant 1, n \geqslant 2$ as follows.

For each $n \geqslant 2$ let $m(n) = [\log_2 n]$, where $[x]$ is the integer part of real x. Then $n \in V_{m(n)}$. Denote by $\phi_{m(n)}(n)_k$ the kth coordinate of the vector $\phi_{m(n)}(n) \in U_{m(n)}$ and for a fixed $s > 0$ put

$$f_n(v) = \begin{cases} \phi_{m(n)}(n)_k (\log n)^{-1+1/\alpha} (\log\log n)^s & \text{if } v \in [a_{k-1}, a_k), 1 \leqslant k \leqslant m(n), \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Let M be a $S\alpha S$ random measure on $(0,1)$ with Borel σ-algebra and Lebesgue control measure. Let $X(n)$ be the corresponded process defined by (1). According to (16)

$$|f_n(v)| \leqslant (\log n)^{-1+1/\alpha} (\log\log n)^s$$

for all $v \in (0,1)$ and, therefore (4) holds for each $\tau \in (0,1)$.

Denote

$$Z_k = [C(\gamma) k (\log(k+1))^\gamma]^{1/\alpha} M([a_{k-1}, a_k)).$$

Then $\{Z_k\}$ is a sequence of i.i.d. $S\alpha S$ random variables. Further, (16) implies that for each $m \geqslant 2$ and each $\omega \in \Omega$ there is an integer $n = n(\omega) \in [2^m, 2^{m+1})$ such that

$$\begin{aligned} X(n,\omega) &= \sum_{k=1}^{m} (\log n)^{-1+1/\alpha} (\log\log n)^s |M((a_{k-1}, a_k))(\omega)| \\ &= C(\gamma)^{-1/\alpha} (\log n)^{-1+1/\alpha} (\log\log n)^s \sum_{k=1}^{m} k^{-1/\alpha} [\log(k+1)]^{-\gamma/\alpha} |Z_k(\omega)|, \end{aligned}$$

which yields

$$\sup_{2^m \leqslant n < 2^{m+1}} |X(n)| \geqslant c m^{-1+1/\alpha} (\log m)^s \sum_{k=1}^{m} k^{-1/\alpha} [\log(k+1)]^{-\gamma/\alpha} |Z_k(\omega)|. \tag{17}$$

Putting

$$S_k = |Z_1| + \cdots + |Z_k| \tag{18}$$

we represent the last sum in the form

$$\sum_{k=1}^{m-1} \left[\frac{1}{k^{1/\alpha} (\log(k+1))^{\gamma/\alpha}} - \frac{1}{(k+1)^{1/\alpha} (\log(k+2))^{\gamma/\alpha}} \right] S_k + \frac{S_m}{m^{1/\alpha} (\log(m+1))^{\gamma/\alpha}}. \tag{19}$$

According to the Strong Law of Large Numbers

$$\lim_{k\to\infty} \frac{S_k}{k} = E|Z_1| := b \quad \text{a.s.}$$

Denoting by Ω_0, $P(\Omega_0) = 1$, the event on which this relation holds, we conclude that for each $\omega \in \Omega_0$ there is $k_0 = k_0(\omega)$ such that $S_k > bk/2$ for all $k > k_0$, and (19) then implies for $m > k_0$

$$m^{-1+1/\alpha}(\log m)^s \sum_{k=1}^{m-1} k^{-1/\alpha}(\log(k+1))^{-\gamma/\alpha}|Z_k(\omega)|$$

$$\geqslant m^{-1+1/\alpha}(\log m)^s \sum_{k=k_0+1}^{m} [k^{-1/\alpha}(\log(k+1))^{-\gamma/\alpha} - (k+1)^{-1/\alpha}(\log(k+2))^{-\gamma/\alpha}]\frac{bk}{2}.$$

Since $\alpha > 1$, the last sum is equivalent to $cm^{1-1/\alpha}(\log m)^{-\gamma/\alpha}$ as $m \to \infty$. Using (17) we see that

$$\sup_m \sup_{2^m \leqslant n < 2^{m+1}} |X(n,\omega)| \geqslant c \liminf_{m\to\infty} (\log m)^{s-\gamma/\alpha} \tag{20}$$

for each $\omega \in \Omega_0$. Now we choose $s > \gamma/\alpha$. For these γ and s the last limit is infinite and, therefore, the process $X(n)$ is unbounded on Ω_0.

The case $\alpha = 1$. We once again fix a $\gamma > 1$ and denote now

$$C(\gamma) = \sum_{j=1}^{\infty} \frac{1}{j\log(j+1)(\log\log(j+2))^\gamma},$$

$$a_0 = 0, \qquad a_k = \frac{1}{C(\gamma)} \sum_{j=1}^{k} \frac{1}{j\log(j+1)(\log\log(j+2))^\gamma}, \quad k = 1,2,\ldots .$$

Using the above notations we put for a fixed $s > 0$

$$f_n(v) = \begin{cases} \phi_{m(n)}(n)_k(\log\log n)^{-1}[\log\log\log(n+3)]^s & \text{if } v \in [a_{k-1}, a_k),\ 1 \leqslant k \leqslant m(n), \\ 0 & \text{otherwise} \end{cases}$$

and choose M to be $S1S$ random measure on $(0,1)$ with Lebesgue control measure. It immediately follows from the definition that (4) holds for each $\tau \in (0,1)$. Denoting

$$Z_k = [C(\gamma)k\log(k+1)(\log\log(k+2))^\gamma]M([a_{k-1}, a_k))$$

we obtain a sequence of i.i.d. $S1S$ random variables and, as above, a bound

$$\sup_{2^m \leqslant n < 2^{m+1}} |X(n)| \geqslant c(\log m)^{-1}(\log\log m)^s \sum_{k=1}^{m} \frac{|Z_k|}{k\log(k+1)(\log\log(k+2))^\gamma}$$

$$\geqslant c(\log m)^{-1}(\log\log m)^s \sum_{k=1}^{m-1} \left[\frac{1}{k\log(k+1)(\log\log(k+2))^\gamma} - \frac{1}{(k+1)\log(k+2)(\log\log(k+3))^\gamma}\right] S_k, \tag{21}$$

where S_k is defined by (18). Because now $E|Z_1| = \infty$, we use the Strong Law of Large Numbers in a different way.

Put

$$Y_k = |Z_k|\mathbf{1}_{\{|Z_k| \leqslant \sqrt{k}\}}, \quad k = 1,2,\ldots .$$

One can easily verify that $c\log(k+1) \leqslant \mu_k := EY_k \leqslant C\log(k+1)$ and $c\sqrt{k} \leqslant \sigma_k^2 := \mathrm{Var}(Y_k) \leqslant C\sqrt{k}$. Hence $\sum_{k=1}^{\infty} \sigma_k^2 k^{-2} < \infty$, which implies that

$$\lim_{n\to\infty}\left(\frac{Y_1+\cdots+Y_k}{k} - \frac{\mu_1+\cdots+\mu_k}{k}\right) = 0 \quad \text{a.s.}$$

(Petrov, 1995, c. 209). Denote by Ω_0, $P(\Omega_0)=1$, the event on which this relation holds and fix $\omega \in \Omega_0$. Then there is an integer $k_0 = k_0(\omega)$ such that for all $k > k_0$

$$\begin{aligned} Y_1(\omega)+\cdots+Y_k(\omega) &\geqslant \mu_1+\cdots+\mu_k - k \geqslant c(\log 2+\cdots+\log(k+1)) - k \\ &= c\log((k+1)!) - k \geqslant c_1 k\log(k+1), \end{aligned}$$

where the last bound follows from Stirling's formula. Since $|Z_k| \geqslant Y_k$ and

$$\begin{aligned} &\frac{1}{k\log(k+1)(\log\log(k+2))^\gamma} - \frac{1}{(k+1)\log(k+2)(\log\log(k+3))^\gamma} \\ &\quad \geqslant \frac{c}{k^2\log(k+1)(\log\log(k+2))^\gamma} \end{aligned}$$

we obtain from (21)

$$\sup_{2^m \leqslant n < 2^{m+1}} |X(n,\omega)| \geqslant c(\log m)^{-1}(\log\log m)^s \sum_{k=k_0+1}^{m-1} \frac{1}{k(\log\log(k+2))^\gamma}. \tag{22}$$

One can easily check that the last sum is equivalent to $c(\log m)(\log\log m)^{-\gamma}$ as $m \to \infty$. So, the last estimates give us

$$\sup_m \sup_{2^m \leqslant n < 2^{m+1}} |X(n)| \geqslant c \liminf_{m\to\infty} (\log\log m)^{s-\gamma}$$

on Ω_0. Choosing $s > \gamma$ we obtain an a.s. unbounded process.

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